Optik - International Journal for Light and Electron Optics 272 (2023) 170215

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.com/locate/ijleo

# On the dynamics of soliton waves in a generalized nonlinear Schrödinger equation

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ARTICLE INFO

*Keywords:*
Generalized nonlinear Schrödinger equation
Complex wave transformation
Modified Jacobi elliptic expansion method
Solitons
Jacobi elliptic structures

ABSTRACT

The main objective of the present paper is to investigate the dynamics of soliton waves in a generalized nonlinear Schrödinger (GNLS) equation. To achieve such a goal, the real and imaginary portions of the GNLS equation are first extracted using a complex wave transformation. Solitons and Jacobi elliptic structures of the governing model, describing the propagation of femtosecond pulses in nonlinear optical fibers, are then constructed through applying the modified Jacobi elliptic expansion method (MJEEM). In the end, by employing a series of 2 and 3D-dimensional numerical representations, it is exposed that the width of bright and dark solitons respectively decreases and increases, while the amplitude of both waves decreases with the increase of nonlinear parameters.

## 1. Introduction

As it is clear to all, solitons play a crucial role in diverse areas of scientific disciplines such as plasma physics, quantum electronics, and nonlinear optics, as they are valuable tools to understand the dynamics of nonlinear phenomena modeled by nonlinear partial differential (NLPD) equations. The nonlinear Schrödinger equation

$$iu_t + \frac{1}{2}u_{tt} + |u|^2 u = 0,$$

is an example of NLPD equations that is used to describe picosecond duration pulses in optical fibers [1]. As pulse duration diminishes, the results of the NLS equation are not reliable, and so the NLS equation must be generalized [1]. To this end, based on a multiple-scale perturbation calculation, Kodama [2] proposed a generalized nonlinear Schrödinger equation as follows

$$iu_t + u_{xx} + |u|^2 u + i\alpha\left(u_{xxx} + \beta_1 |u|^2 u_x + \beta_2 u^2 u_x^*\right) = 0, \tag{1}$$

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https://doi.org/10.1016/j.ijleo.2022.170215
Received 24 October 2022; Received in revised form 7 November 2022; Accepted 7 November 2022
Available online 9 November 2022
0030-4026/

where $u^*$ is the complex conjugate of $u$, $u_{xx}$ is the group velocity dispersion, $|u|^2u$ is the Kerr law nonlinearity, $u_{xxx}$ is the third-order dispersion, and $|u|^2u_x$ and $u^2u_x^*$ are nonlinear terms. Potasek and Tabor [1] derived soliton solutions of Eq. (1) using the ansatz methods. Very recently, Kumari et al. [3] employed the generalized Jacobi elliptic method to retrieve the Jacobi elliptic-type solutions of the GNLS equation.

Nowadays, the handling of symbolic computations has become more convenient with the advancement of different computer algebra systems like MATHEMATICA, MAPLE, etc. Such a development has made the finding of soliton solutions of NLPD equations easier. In recent decades, numerous influential methods such as the exponential method [4–9], the Kudryashov method [10–15], and the modified Jacobi elliptic expansion method [16–22] have been proposed and used in the literature. The main objective of the present paper is to derive solitons and Jacobi elliptic structures of the generalized nonlinear Schrödinger equation through applying the modified Jacobi elliptic expansion method and investigating the dynamics of its soliton waves. The MJEEM, as a worthwhile technique in dealing with NLPD equations, plays an important role in many recently published papers. For example, Khalil et al. [19] conducted a study on soliton waves of the coupled system of nonlinear Biswas–Milovic equation using the MJEEM. El-Sheikh et al. [20] explored the propagation of capillary-gravity waves on the shallow water surface by applying the MJEEM to a nonlinear Boussinesq-type equation of high-order. Hosseini and Mirzazadeh [21] applied the MJEEM to derive soliton and other solutions of the chiral nonlinear Schrödinger equation. In another study, the authors [22] investigated the soliton dynamics in waveguides by employing the MJEEM to the Fokas–Lenells equation.

The organization of the current work is as follows: In Section 2, the MJEEM is described in brief. In Section 3, solitons and Jacobi elliptic structures of the generalized nonlinear Schrödinger equation are acquired by exerting the MJEEM. In Section 4, by applying several 2 and 3D-dimensional numerical representations, it is exposed that the width of bright and dark solitons respectively decreases and increases, while the amplitude of both waves decreases with the increase of nonlinear parameters. A detailed summary of the achievements is given in the last section.

## 2. The method along with its details

The present section aims to review the fundamental of the MJEEM. To this end, let's consider

$$F(u, u', u'', \ldots) = 0, \quad ' = \frac{d}{d\epsilon}, \tag{2}$$

as a nonlinear ordinary differential equation (NLODE).

Based on [16], the following series is chosen as the solution of Eq. (2).

$$u(\epsilon) = c_0 + \sum_{i=1}^{N} \left( \frac{J(\epsilon)}{1 + J^2(\epsilon)} \right)^{i-1} \left( c_i \frac{J(\epsilon)}{1 + J^2(\epsilon)} + d_i \frac{1 - J^2(\epsilon)}{1 + J^2(\epsilon)} \right), \quad c_N \text{ or } d_N \neq 0, \tag{3}$$

where $c_0$, $c_i$, and $d_i$ $(1 \le i \le N)$ are unknowns, $N$ is the number of balance, and $J(\epsilon)$ as a Jacobi elliptic function satisfies

$$(J'(\epsilon))^2 = D + EJ^2(\epsilon) + FJ^4(\epsilon). \tag{4}$$

The Jacobi elliptic Eq. (4) depending on $D$, $E$, and $F$ admits the following exact solutions (See Table 1):

By setting the series (3) in Eq. (2) and applying symbolic computation, a nonlinear algebraic system is acquired whose solution yields exact solutions of Eq. (2).

The Jacobi elliptic functions have the following features:

1. $\mathrm{sn}^2(\epsilon) + \mathrm{cn}^2(\epsilon) = 1$,
2. $\mathrm{sn}(\epsilon) = \mathrm{sn}(\epsilon, m) \to \tanh(\epsilon)$ when $m \to 1$,
3. $\mathrm{ns}(\epsilon) = (\mathrm{sn}(\epsilon, m))^{-1} \to \coth(\epsilon)$ when $m \to 1$.

## 3. The governing model: Its solitons and Jacobi elliptic structures

In the current section, solitons and Jacobi elliptic structures of the generalized nonlinear Schrödinger equation are derived by employing the MJEEM. To this end, the authors are interested in applying the following transformation

**Table 1**
Eq. (4) and its Jacobi elliptic function solutions.

| No. | $D$ | $E$ | $F$ | $J(\epsilon)$ |
|---|---|---|---|---|
| 1 | $1$ | $-(m^2+1)$ | $m^2$ | $\mathrm{sn}(\epsilon)$ |
| 2 | $1-m^2$ | $2m^2-1$ | $-m^2$ | $\mathrm{cn}(\epsilon)$ |
| 3 | $m^2$ | $-(m^2+1)$ | $1$ | $\mathrm{ns}(\epsilon)$ |
| 4 | $-m^2$ | $2m^2-1$ | $1-m^2$ | $\mathrm{nc}(\epsilon)$ |

$$u(x,t)=U(\epsilon)e^{i(\kappa_2x-\nu_2t)},\quad \epsilon=\kappa_1x-\nu_1t,$$

to the GNLS equation. It is found that

$$u_t=-(i\nu_2U(\epsilon)+\nu_1U'(\epsilon))e^{i(\kappa_2x-\nu_2t)},$$

$$u_{xx}=\left(2i\kappa_1\kappa_2U'(\epsilon)-\kappa_2^2U(\epsilon)+\kappa_1^2U''(\epsilon)\right)e^{i(\kappa_2x-\nu_2t)},$$

$$|u|^2u=U^3(\epsilon)e^{i(\kappa_2x-\nu_2t)},$$

$$u_{xxx}=\left(-i\kappa_2^3U(\epsilon)+3i\kappa_1^2\kappa_2U''(\epsilon)-3\kappa_1\kappa_2^2U'(\epsilon)+\kappa_1^3U'''(\epsilon)\right)e^{i(\kappa_2x-\nu_2t)},$$

$$|u|^2u_x=U^2(\epsilon)(i\kappa_2U(\epsilon)+\kappa_1U'(\epsilon))e^{i(\kappa_2x-\nu_2t)},$$

$$u^2u_x^*=U^2(\epsilon)(-i\kappa_2U(\epsilon)+\kappa_1U'(\epsilon))e^{i(\kappa_2x-\nu_2t)}.$$

The above relations together with the governing model yield

$$\kappa_1^2(-3\alpha\kappa_2+1)U''(\epsilon)+\left(\alpha\kappa_2^3-\kappa_2^2+\nu_2\right)U(\epsilon)+(1-\kappa_2(\beta_1-\beta_2)\alpha)U^3(\epsilon)=0, \tag{5}$$

$$\alpha\kappa_1(\beta_1+\beta_2)U'(\epsilon)U^2(\epsilon)+\left(-3\alpha\kappa_1\kappa_2^2+2\kappa_1\kappa_2-\nu_1\right)U'(\epsilon)+\alpha\kappa_1^3U'''(\epsilon)=0.$$

After integrating the second equation w.r.t. $\epsilon$, we find

$$\alpha\kappa_1^3U''(\epsilon)+\left(-3\alpha\kappa_1\kappa_2^2+2\kappa_1\kappa_2-\nu_1\right)U(\epsilon)+\frac{1}{3}\alpha\kappa_1(\beta_1+\beta_2)U^3(\epsilon)=0. \tag{6}$$

Comparing the corresponding terms in Eqs. (5) and (6) results in

$$\kappa_2=-\frac{\alpha\kappa_1-1}{3\alpha},$$

$$\nu_1=-\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2},$$

$$\alpha=-\frac{-3+\beta_1-\beta_2}{2\kappa_1\beta_2}.$$

Therefore, it is enough to solve Eq. (5) with the above conditions.

Owing to Eq. (5) and its special terms i.e. $U''(\epsilon)$ and $U^3(\epsilon)$, we find $N=1$. It recommends a nontrivial structure as follows

$$U(\epsilon)=c_0+c_1\frac{J(\epsilon)}{1+J^2(\epsilon)}+c_2\frac{1-J^2(\epsilon)}{1+J^2(\epsilon)},\quad c_2=d_1, \tag{7}$$

which $c_0$, $c_1$, and $c_2$ are determined in the solution process and $J(\epsilon)$ is a Jacobi elliptic function. By substituting the nontrivial structure (7) into Eq. (5) and applying symbolic computations, a nonlinear system of algebraic type is acquired whose solution yields:

**Case 1.** For $D=1$, $E=-(m^2+1)$, and $F=m^2$, the results are

I. $c_0=0$, $c_1=\pm4\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_2=0$, $\nu_2=-\left(\kappa_1^2\left(217\beta_1^3-651\beta_1^2\beta_2+639\beta_1\beta_2^2-221\beta_2^3-1953\beta_1^2+3906\beta_1\beta_2-1917\beta_2^2+5859\beta_1-5859\beta_2-5859\right)\right)/\left(54\beta_2\left(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9\right)\right)$, $m=1$,

II. $c_0=0$, $c_1=0$, $c_2=\pm2\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2=\left(\kappa_1^2\left(107\beta_1^3-321\beta_1^2\beta_2+333\beta_1\beta_2^2-103\beta_2^3-963\beta_1^2+1926\beta_1\beta_2-999\beta_2^2+2889\beta_1-2889\beta_2-2889\right)\right)/\left(54\beta_2\left(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9\right)\right)$, $m=1$,

III. $c_0=0$, $c_1=\pm2\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_2=\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2=-\left(\kappa_1^2\left(55\beta_1^3-165\beta_1^2\beta_2+153\beta_1\beta_2^2-59\beta_2^3-495\beta_1^2+990\beta_1\beta_2-459\beta_2^2+1485\beta_1-1485\beta_2-1485\right)\right)/\left(54\beta_2\left(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9\right)\right)$, $m=1$.

Since $J(\epsilon)=\mathrm{sn}(\epsilon)$ and $\mathrm{sn}(\epsilon,1)\to\tanh(\epsilon)$, thus, the exact solutions of the GNLS equation are as

$$u_{1,2}(x,t)=\pm4\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1\frac{\tanh\left(\kappa_1x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\tanh^2\left(\kappa_1x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2t\right)},\ \nu_2\text{ in I},$$

$$u_{3,4}(x,t)=\pm 2\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\tanh^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\tanh^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)},\nu_2 \text{ in } II,$$

$$u_{5,6}(x,t)=(\pm 2\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1\frac{\tanh\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\tanh^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}$$

$$+\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\tanh^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\tanh^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)})e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)},\nu_2 \text{ in } III.$$

**Case 2.** For $D=1-m^2$, $E=2m^2-1$, and $F=-m^2$, the results are

I. $c_0=\pm\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_1=0$, $c_2=\pm\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2=(\kappa_1^2(133\beta_1^3-399\beta_1^2\beta_2+423\beta_1\beta_2^2-125\beta_2^3-1197\beta_1^2+2394\beta_1\beta_2-1269\beta_2^2+3591\beta_1-3591\beta_2-3591))/(108\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)),m=\frac{1}{2}$,

II. $c_0=\pm\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_1=0$, $c_2=\pm\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2=(\kappa_1^2(133\beta_1^3-399\beta_1^2\beta_2+423\beta_1\beta_2^2-125\beta_2^3-1197\beta_1^2+2394\beta_1\beta_2-1269\beta_2^2+3591\beta_1-3591\beta_2-3591))/(108\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)),m=-\frac{1}{2}$.

Since $J(\epsilon)=\mathrm{cn}(\epsilon)$, thus, the exact solutions of the GNLS equation are as follows

$$u_{7,8}(x,t)=\left(\pm\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\pm\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\mathrm{cn}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t,\frac{1}{2}\right)}{1+\mathrm{cn}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t,\frac{1}{2}\right)}\right)e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)},\nu_2 \text{ in } I,$$

$$u_{9,10}(x,t)=\left(\pm\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\pm\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\mathrm{cn}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t,-\frac{1}{2}\right)}{1+\mathrm{cn}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t,-\frac{1}{2}\right)}\right)e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)},\nu_2 \text{ in } II.$$

**Case 3.** For $D=m^2$, $E=-(m^2+1)$, and $F=1$, the results are

I. $c_0=0$, $c_1=\pm 4\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_2=0$, $\nu_2=-(\kappa_1^2(217\beta_1^3-651\beta_1^2\beta_2+639\beta_1\beta_2^2-221\beta_2^3-1953\beta_1^2+3906\beta_1\beta_2-1917\beta_2^2+5859\beta_1-5859\beta_2-5859))/(54\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)),m=1$,

II. $c_0=0$, $c_1=0,c_2=\pm 2\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2=(\kappa_1^2(107\beta_1^3-321\beta_1^2\beta_2+333\beta_1\beta_2^2-103\beta_2^3-963\beta_1^2+1926\beta_1\beta_2-999\beta_2^2+2889\beta_1-2889\beta_2-2889))/(54\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)),m=1$,

III. $c_0=0$, $c_1=\pm 2\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_2=\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2=-(\kappa_1^2(55\beta_1^3-165\beta_1^2\beta_2+153\beta_1\beta_2^2-59\beta_2^3-495\beta_1^2+990\beta_1\beta_2-459\beta_2^2+1485\beta_1-1485\beta_2-1485))/(54\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)),m=1$.

Since $J(\epsilon)=\mathrm{ns}(\epsilon)$ and $\mathrm{ns}(\epsilon,1)\to\coth(\epsilon)$, hence, the exact solutions of the GNLS equation are as

$$u_{11,12}(x,t)=\pm 4\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1\frac{\coth\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\coth^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)},\nu_2 \text{ in } I,$$

$$u_{13,14}(x,t)=\pm 2\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\coth^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\coth^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)},\nu_2 \text{ in } II,$$

$$u_{15,16}(x,t)=(\pm 2\sqrt{-6(\beta_1+\beta_2)^{-1}}\kappa_1\frac{\coth\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\coth^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}$$

$$+\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\coth^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}{1+\coth^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t\right)}\Bigg)e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)}, \nu_2 \text{ in } III.$$

**Case 4.** For $D = -m^2$, $E = 2m^2-1$, and $F = 1-m^2$, the results are

I. $c_0 = \mp\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_1 = 0$, $c_2 = \pm\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2 = (\kappa_1^2(133\beta_1^3-399\beta_1^2\beta_2+423\beta_1\beta_2^2-125\beta_2^3-1197\beta_1^2+2394\beta_1\beta_2-1269\beta_2^2+3591\beta_1-3591\beta_2-3591))/(108\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)), m=\frac{1}{2}$,

II. $c_0 = \mp\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $c_1 = 0$, $c_2 = \pm\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1$, $\nu_2 = (\kappa_1^2(133\beta_1^3-399\beta_1^2\beta_2+423\beta_1\beta_2^2-125\beta_2^3-1197\beta_1^2+2394\beta_1\beta_2-1269\beta_2^2+3591\beta_1-3591\beta_2-3591))/(108\beta_2(\beta_1^2-2\beta_1\beta_2+\beta_2^2-6\beta_1+6\beta_2+9)), m=-\frac{1}{2}$.

Since $J(\epsilon) = \mathrm{nc}(\epsilon)$, therefore, the exact solutions of the GNLS equation are as follows

$$u_{17,18}(x,t) = \left(\mp\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1 \pm \sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\mathrm{nc}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t, \frac{1}{2}\right)}{1+\mathrm{nc}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t, \frac{1}{2}\right)}\right)e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)}, \nu_2 \text{ in } I,$$

$$u_{19,20}(x,t) = \left(\mp\frac{1}{2}\sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1 \pm \sqrt{6}\sqrt{(\beta_1+\beta_2)^{-1}}\kappa_1\frac{1-\mathrm{nc}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t, -\frac{1}{2}\right)}{1+\mathrm{nc}^2\left(\kappa_1 x+\frac{8\alpha^3\kappa_1^3+27\alpha^2\nu_2-6\alpha\kappa_1-2}{27\alpha^2}t, -\frac{1}{2}\right)}\right)e^{i\left(-\frac{\alpha\kappa_1-1}{3\alpha}x-\nu_2 t\right)}, \nu_2 \text{ in } II.$$

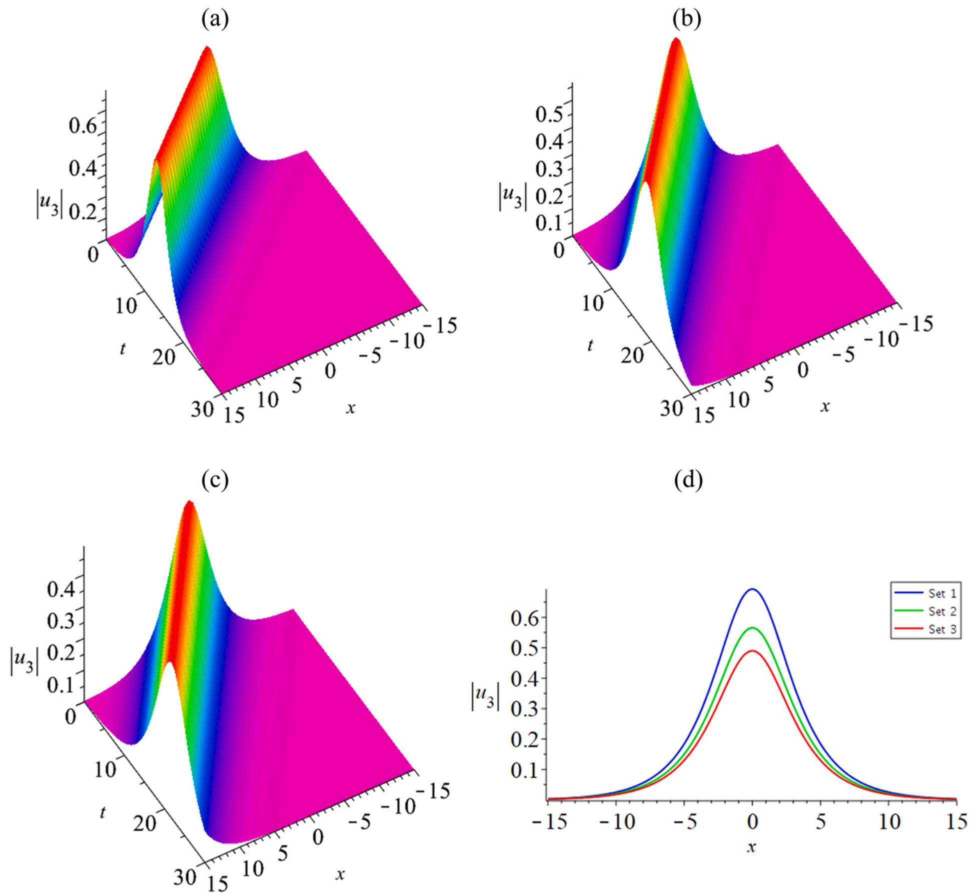


**Fig. 1.** The third bright soliton for (a) Set 1 ($\{\beta_1=1, \beta_2=1, \kappa_1=0.2\}$), (b) Set 2 ($\{\beta_1=2, \beta_2=1, \kappa_1=0.2\}$), (c) Set 3 ($\{\beta_1=1, \beta_2=3, \kappa_1=0.2\}$), and (d) all sets when $t=0$.

## 4. Graphical representations and discussion

In Section 3, a group of exact solutions of the GNLS equation in the presence of the group velocity dispersion, the third-order dispersion, and different nonlinearities was retrieved by employing the MJEEM. In the current section, the effect of the nonlinear parameters on the dynamical features of bright and dark solitons is investigated. For this goal, first to examine the effect of $\beta_1$ and $\beta_2$ on the dynamical features of the bright soliton, the following sets have been chosen:

$$\text{Set1} : \{\beta_1 = 1, \beta_2 = 1, \kappa_1 = 0.2\},$$

$$\text{Set2} : \{\beta_1 = 2, \beta_2 = 1, \kappa_1 = 0.2\},$$

$$\text{Set2} : \{\beta_1 = 1, \beta_2 = 3, \kappa_1 = 0.2\}.$$

From Fig. 1, it is observed that the width of the bright soliton decreases with the increase of $\beta_1$ or $\beta_2$. Furthermore, by increasing $\beta_1$ or $\beta_2$, the amplitude of the bright soliton decreases.

In order to investigate the effect of $\beta_1$ and $\beta_2$ on the dynamical structures of the dark soliton, the following sets have been selected:

$$\text{Set1} : \{\beta_1 = 1.5, \beta_2 = 1.5, \kappa_1 = 0.1\},$$

$$\text{Set2} : \{\beta_1 = 2.5, \beta_2 = 1.5, \kappa_1 = 0.1\},$$

$$\text{Set2} : \{\beta_1 = 1.5, \beta_2 = 3.5, \kappa_1 = 0.1\}.$$

By looking at Fig. 2, it is found that the width of the dark soliton increases with the increase of $\beta_1$ or $\beta_2$. Additionally, by increasing $\beta_1$ or $\beta_2$, the amplitude of the dark soliton decreases.

Based on the results presented, it can be deduced that we are capable of controlling the dynamical evolution of soliton waves in the

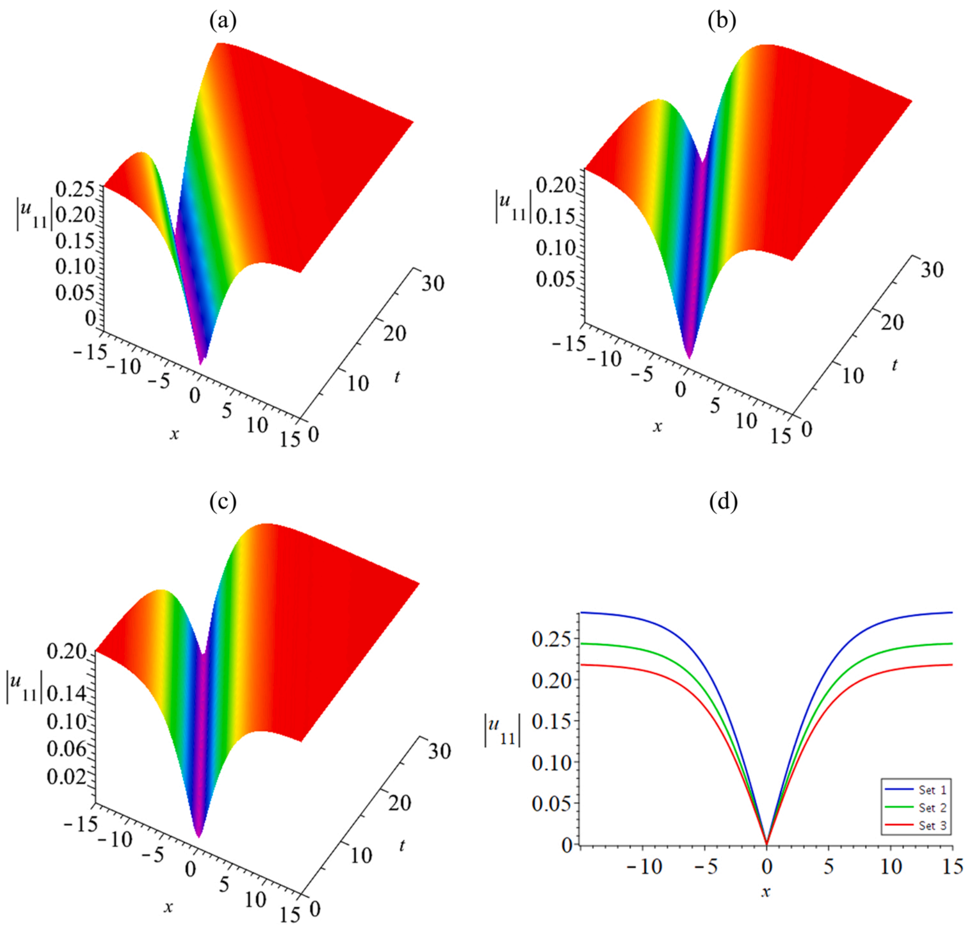


**Fig. 2.** The eleventh dark soliton for (a) Set 1 ($\{\beta_1 = 1.5, \beta_2 = 1.5, \kappa_1 = 0.1\}$), (b) Set 2 ($\{\beta_1 = 2.5, \beta_2 = 1.5, \kappa_1 = 0.1\}$), (c) Set 3 ($\{\beta_1 = 1.5, \beta_2 = 3.5, \kappa_1 = 0.1\}$), and (d) all sets when $t = 0$.

GNLS equation involving the group velocity dispersion, the third-order dispersion, and different nonlinearities.

## 5. Conclusion

The current paper investigated the dynamics of soliton waves in a generalized nonlinear Schrödinger equation, describing the propagation of femtosecond pulses in nonlinear optical fibers. To this end, a complex wave transformation was first utilized to arrive at the reduced form of the governing model. Solitons and Jacobi elliptic structures of the GNLS equation were then retrieved by employing the MJEEM. Several graphical representations were considered to investigate the effect of the nonlinear parameters on the dynamical features of bright and dark solitons. It was found that

I. The width of the bright soliton decreases with the increase of $\beta_1$ or $\beta_2$.
II. The width of the dark soliton increases with the increase of $\beta_1$ or $\beta_2$.
III. The amplitude of bright and dark solitons decreases with the increase of $\beta_1$ or $\beta_2$.

The results of the present paper give ways of controlling the dynamical evolution of soliton waves in the GNLS equation involving the group velocity dispersion, the third-order dispersion, and different nonlinearities. The authors are interested in applying other methods [23–43] to the governing model for deriving its other wave patterns.

## Declaration of Competing Interest

No conflict of interest exits in the submission of this manuscript, and manuscript is approved by all authors for publication.

## Data Availability

No data was used for the research described in the article.

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